Journal of Computational Physics **159,** 440–447 (2000)
doi:10.1006/jcph.2000.6423, available online at http://www.idealibrary.com on IDEAL

NOTE

Instability of the Filtering Method for Vlasov's Equation

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Received March 9, 1999; revised October 19, 1999

1. INTRODUCTION

The earliest numerical methods introduced to solve the Vlasov–Poisson system were polynomial expansions [1]. In these methods, the position dependence is usually expanded in Fourier modes and the velocity dependence is treated either through Fourier modes [2–6] or Hermite polynomials [7–11]. Then splitting schemes appeared. In those schemes the initial Vlasov equations split in two partial differential equations, one in x, t the other in v, t. These equations must be solved alternatively [1]. A simple way of solving the splited equations is to use Fourier transform alternatively for both x and v subspaces [12, 13]. The tendency of the distribution function $f(x, v, t)$ to develop steep gradients in phase space ("the filamentation") inhibits the numerical solution to the Vlasov–Poisson system [13]. In order to ward off this problem Klimas has introduced a smoothed Fourier–Fourier method [16]. This method consists of convolving the original distribution function with a Gaussian distribution function and then solving the new system with a transformed splitting algorithm. Unfortunately, a second-order term appears in the new equation.

In this work, we study how this term affects the numerical resolution. In particular we prove that instability occurs in the linear version of the Vlasov equation obtained by considering only free noninteracting particles. We prove also that the use of Fourier–Fourier transform is a fundamental requirement for solving numerically this new equation. We point out an important property, which is not completely clarified in [16], concerning the filtered distribution function in the transformed space. The paper is organized as follows. In the second section we define the mathematical model. In Section 3 we prove the instability of the smoothed equation with respect to perturbations. Section 4 is devoted to the need of

0021-9991/00 $35.00

using Fourier–Fourier transforms to obtain a stable splitting scheme. Our conclusions are given in Section 5.

2. THE MATHEMATICAL MODEL

The evolution of a one-dimensional electron plasma in a periodic box can be described by the normalized Vlasov–Poisson system.

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + E(x,t)\frac{\partial f}{\partial v} = 0, \tag{1}$$

$$\frac{\partial E}{\partial x} = \int f(x,v,t)\,\mathrm{d}v - 1, \qquad \frac{1}{L}\iint f(x,v,t)\,\mathrm{d}x\,\mathrm{d}v = 1, \tag{2}$$

where $f(x,v,t)$ denotes the electron distribution function, $E(x,t)$ is the electric field, and L is the length of the periodic spatial box. In these units t is normalized to the inverse of plasma frequency w_p, v to the thermal velocity v_{th}, and x to the Debye length λ_{D}.

The idea to use a splitting algorithm in time to integrate the Vlasov equation (1) was introduced first in [2]. As it is difficult to distinguish between the mathematical filamentation and the numerical noise, a method of filtering was introduced in [16]. Its philosophy consists of a convolution of the distribution function f by a Gaussian filter in the variable v to obtain the smoothed function $\bar{f}$,

$$\bar{f}(x,v,t) = \int F(v-u)\,f(x,u,t)\,\mathrm{d}u, \tag{3}$$

where

$$F(v) = \frac{1}{\sqrt{2\pi}\,v_0}e^{-(v/v_0)^2/2}, \tag{4}$$

and v_0 is a constant parameter giving the width of the Gaussian filter in thermal velocity units. We remark that $\bar{E} = E$ since

$$\int f\,\mathrm{d}v = \int \bar{f}\,\mathrm{d}v. \tag{5}$$

Now, in order to obtain a new equation on $\bar{f}$ we compute as in [16]

$$\frac{\partial \bar{f}}{\partial t} = \int F(v-u)\frac{\partial f}{\partial t}\,\mathrm{d}u, \tag{6}$$

$$\begin{aligned}
v\frac{\partial \bar{f}}{\partial x} &= v\int F(v-u)\frac{\partial f}{\partial x}\,\mathrm{d}u \\
&= \int (v-u)F(v-u)\frac{\partial f}{\partial x}\,\mathrm{d}u + \int F(v-u)u\frac{\partial f}{\partial x}\,\mathrm{d}u \\
&= \left[v_0^2 F(v-u)\frac{\partial f}{\partial x}\right]_{-\infty}^{+\infty} - v_0^2\int F(v-u)\frac{\partial^2 f}{\partial x\,\partial u}\,\mathrm{d}u + \int F(v-u)u\frac{\partial f}{\partial x}\,\mathrm{d}u \\
&= -v_0^2\frac{\partial^2 \bar{f}}{\partial x\,\partial v} + \int F(v-u)u\frac{\partial f}{\partial x}\,\mathrm{d}u,
\end{aligned} \tag{7}$$

$$\frac{\partial \bar{f}}{\partial v} = -\int \frac{(v-u)}{v_0^2} F(v-u)\, f(x,u,t)\,\mathrm{d}u$$
$$= \int F(v-u)\frac{\partial f}{\partial u}\,\mathrm{d}u. \tag{8}$$

Combining, (6), (7), and (8), we obtain that $\bar{f}$ solves

$$\frac{\partial \bar{f}}{\partial t} + v\frac{\partial \bar{f}}{\partial x} + \bar{E}(x,t)\frac{\partial \bar{f}}{\partial v} = -v_0^2 \frac{\partial^2 \bar{f}}{\partial x \partial v}, \tag{9}$$

$$\frac{\partial \bar{E}}{\partial x} = \int \bar{f}(x,v,t)\,\mathrm{d}v - 1. \tag{10}$$

The aim of the present paper is to compare the stability properties with respect to perturbations of Eqs. (1) and (9). The conclusion we get is that the solutions to (9) can be obtained only by the use of Fourier transforms and so are very sensitive to perturbations. Consequently we have to be extremely careful when using such a method for numerical computation, in the case of general initial conditions.

Since the filamentation process is associated to the free streaming term $v\frac{\partial f}{\partial x}$, it is sufficient to consider the free streaming problem, dropping into (9) the term $\bar{E}(x,t)\frac{\partial \bar{f}}{\partial v}$. Thus, let us only consider the equation

$$\begin{cases} \frac{\partial g}{\partial t} + v\frac{\partial g}{\partial x} = -v_0^2 \frac{\partial^2 g}{\partial x \partial v}, \\ g(x,v,0) = g_0(x,v). \end{cases} \tag{11}$$

In order to solve exactly Eq. (11), let us define the double Fourier transform $\tilde{g}$ of g both in x and v by

$$\tilde{g}(m,\nu,t) = \frac{1}{L}\int_{x=0}^{L}\int_{v\in\mathbb{R}} e^{-i\left(m\frac{2\pi}{L}x + \nu v\right)} g(x,v,t)\,\mathrm{d}x\,\mathrm{d}v, \tag{12}$$

Introducing (12) in (11), we obtain

$$\frac{\partial \tilde{g}}{\partial t} - k_0 m \frac{\partial \tilde{g}}{\partial \nu} = v_0^2 k_0 m \nu \tilde{g}, \tag{13}$$

where k_0 is the fundamental wave number $k_0 = \frac{2\pi}{L}$.

Now let us study the Cauchy problem, which consists of solving Eq. (13) with initial condition

$$\tilde{g}(m,\nu,0) = \tilde{g}_0(m,\nu). \tag{14}$$

The solution of system (13)–(14) is given by

$$\tilde{g}(m,\nu,t) = \tilde{g}_0(m,\nu + mk_0 t)e^{v_0^2 m k_0 \nu t} e^{\frac{1}{2}v_0^2 m^2 k_0^2 t^2}. \tag{15}$$

Then, in order to obtain a solution to (11), we need to find a function g such that its Fourier transform is $\tilde{g}$ defined in (15). If $\tilde{g}_0(m,\nu)$ is an arbitrary function, we observe that, asymptotically, if ν and m have the same sign then the term $e^{v_0^2 m k_0 \nu t} e^{\frac{1}{2} v_0^2 m^2 k_0^2 t^2}$ in (15) tends exponentially to infinity, and therefore there is no function having $\tilde{g}$ as the Fourier transform. Consequently there is no well behaved solution to (11) if g_0 is an arbitrary function [14]. On the contrary, let the initial distribution function g_0 take the form

$$g_0(x, v) = f_0(x, v) \times F(v); \tag{16}$$

then

$$\tilde{g}_0(m, \nu) = \tilde{f}_0(m, \nu) e^{-\frac{1}{2} v_0^2 \nu^2}. \tag{17}$$

Hence, we get from (15)

$$\tilde{g}(m, \nu, t) = \tilde{f}_0(m, \nu + m k_0 t) e^{-\frac{1}{2} v_0^2 \nu^2}, \tag{18}$$

or equivalently

$$\tilde{f}(m, \nu, t) = \tilde{f}_0(m, \nu + m k_0 t) = \tilde{g}(m, \nu, t) \mathbf{e}^{\frac{1}{2} v_0^2 \nu^2}, \tag{19}$$

which is the solution to the Vlasov equation. Of course, at this level, equation

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = 0 \tag{20}$$

and Eq. (11) are equivalent, but this may no longer be the case when introducing discrete approximation. By these formulas we see indeed that the fact that g_0 has the form (16) is crucial, and, as we shall see in Section 3, we have to keep this property for all times in discrete approximation.

Our purpose is to understand what does the filtering method. As a first point we have to realize that the filamentation is a physical process and the details of $f(x, v, t)$; i.e., the high ν components must be kept and play a crucial role in situations like in the echo problem [15]. The multiplication of these large ν components by $\mathbf{e}^{-\frac{1}{2} v_0^2 \nu^2}$ neither bring nor destroy the needed information, but we should be concerned that a too large ν_0 can introduce too small $\tilde{g}(m, \nu)$ exposed to destruction by roundoff error.

It must be noticed that many checks of numerical methods use the linear Landau damping as a test case. It is a misleading check. The perturbed part of the distribution function enter only through its components $\nu = 0$ (the perturbed density). All the subtle nonlinear phenomena are hidden in $f(m, \nu, t)$. Notice that $f(m, \nu = 0, t) = \overline{f}(m, \nu = 0, t)$, which is the only value needed for the linear Landau damping problem.

3. STABILITY

It is interesting to investigate the stability of (11) with respect to perturbations. For that purpose, we compare the exact solution to (11), which can be written as

$$g(t) = S(t) g(0), \tag{21}$$

with $S(t)$ the resolution operator, which can be expressed by (15), and an approximate solution h_n computed by

$$h_{n+1} = A(\Delta t)h_n, \tag{22}$$

with $A(\Delta t)$ an approximate resolution operator, and $h_0 = g(0)$. At a fixed time $T = n\Delta t$, we assume that there is a slight difference between h_n and g as

$$h_n = g(T) + \delta g. \tag{23}$$

Then, in the next step, since the operator S is linear, the difference between h_{n+1} and g takes the following form:

$$h_{n+1} - g(t) = (A(\Delta t) - S(\Delta t))h_n + S(\Delta t)\delta g. \tag{24}$$

For this difference to be small, we need both terms in the right-hand side of (24) to be small. The first one depends on the way A approaches S, but for the second one, as we discussed before, $\tilde{\delta g}$ needs to be small with respect to $e^{-v_0^2 v^2/2}$. Therefore, a necessary condition for the approximate method to be stable is that the operator $A(\Delta t)$ preserves the exponential decrease at infinity of the Fourier transform. Generally, this property is very difficult to obtain unless $A(\Delta t)$ is defined itself by Fourier transform. Moreover, it might be lost when taking into account the acceleration term due to the electric field.

4. NEED TO USE FOURIER–FOURIER TRANSFORM

In the following, we show that the resolution of Eq. (9) can be performed only by the use of Fourier transform (without the term $E\frac{\partial f}{\partial v}$). It will be proved that the direct (without Fourier) resolution of Eq. (11) leads to an unstable heat equation. Equation (11) is a second-order linear partial differential equation. It can be solved by Fourier–Fourier transform, but let us try a splitting method as follows:

$$\frac{\partial g}{\partial t} + v\frac{\partial g}{\partial x} = 0, \tag{25}$$

$$\frac{\partial g}{\partial t} + v_0^2\frac{\partial^2 g}{\partial x \partial v} = 0. \tag{26}$$

This system represents a linear transport Eq. (25) and a second-order parabolic equation in a noncanonical form.

The solution of the transport Eq. (25) is given by

$$g(x, v, t) = g(x - vt, v, 0). \tag{27}$$

In order to solve Eq. (26) we introduce the change of variables

$$\begin{cases} 2x = x_1 - y_1, \\ 2v = x_1 + y_1. \end{cases} \tag{28}$$

Introducing the relation (28) into Eq. (26), we obtain the canonical form

$$\frac{\partial g}{\partial t} - v_0^2\frac{\partial^2 g}{\partial y_1^2} + v_0^2\frac{\partial^2 g}{\partial x_1^2} = 0. \tag{29}$$

We have obtained a canonical linear partial differential equation, which can be solved a priori by a splitting method as follows:

$$\frac{\partial g}{\partial t} - v_0^2 \frac{\partial^2 g}{\partial y_1^2} = 0, \tag{30}$$

$$\frac{\partial g}{\partial t} + v_0^2 \frac{\partial^2 g}{\partial x_1^2} = 0. \tag{31}$$

The difference between the last equations reside in their stability. It is well known that Eq. (30) is stable but Eq. (31), called the retrograde heat equation, is unstable with respect to perturbations. Consequently a solution to Eq. (11) by the splitting method (25)–(26) is unstable without Fourier.

Now we remark that the stability of partial differential equations depends mainly on their highest-order terms. Therefore, since we have seen above that Eq. (26) is unstable, the resolution of Eq. (11) is also unstable. Hence we must not separate the terms $v\frac{\partial g}{\partial x}$ and $v_0^2 \frac{\partial^2 g}{\partial x \partial v}$, and we must be very careful in the treatment of these two terms. As shown in Sections 2 and 3, this can be achieved only by the use of Fourier–Fourier transform. In this case the filtering of the initial distribution function becomes a simple multiplication which serves to damp high wavelengths, as we have seen in Section 2. That operation hides but does not remove the filamentation.

In order to understand what happens in the filtering method, let us give a rough caricature of the instability. We consider the trivial equation

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) = 0, \\ u(0, x) = u^0(x), \end{cases} \tag{32}$$

and we consider its convolution by a time-dependent gaussian function,

$$v(t, x) = u(t, x) * \frac{1}{\sqrt{4\pi\sigma(T-t)}} \mathbf{e}^{-x^2/4\sigma(T-t)}. \tag{33}$$

It is easy to see that then v solves

$$\begin{cases} \frac{\partial v}{\partial t} + \sigma \frac{\partial^2 v}{\partial x^2} = 0, \\ v(0, x) = v^0(x). \end{cases} \tag{34}$$

As in the Vlasov case, it is necessary that v^0 has the form of a convolution in order that (34) has a solution.

LEMMA 1. *Assume that v solves for some $\sigma > 0$*

$$\frac{\partial v}{\partial t} + \sigma \frac{\partial^2 v}{\partial x^2} = 0, \qquad (t, x) \in]0, T[\times \mathbb{R}.$$

Then $v(0)$ has the form

$$v(0) = w * \frac{1}{\sqrt{4\pi\sigma T}} \mathbf{e}^{-x^2/4\sigma T}$$

for some function w.

Proof. Define $z(t,x) = v(T-t,x)$ and $w(x) = v(x,T)$. Then z solves the usual heat equation

$$\begin{cases} \frac{\partial z}{\partial t} - \sigma \frac{\partial^2 z}{\partial x^2} = 0, \\ z(0,x) = w(x). \end{cases} \tag{35}$$

It is well-known that the unique solution to (35) is given by the kernel of the heat equation,

$$z(t,x) = w(x) \times \frac{1}{\sqrt{4\pi\sigma t}} \mathbf{e}^{-x^2/4\sigma t}, \tag{36}$$

and in particular

$$v(0,x) = z(T,x) = w(x) \times \frac{1}{\sqrt{4\pi\sigma T}} \mathbf{e}^{-x^2/4\sigma T}. \tag{37}$$

Therefore, Eq. (34) has no solution, unless $v(0,x)$ has the form (37), but it is difficult to keep this form when introducing perturbations. It is well known that the backward heat Eq. (34) is very unstable with respect to perturbations. The only way to solve Eq. (34) is to use Fourier transform, in which case we obtain

$$\frac{\partial \tilde{v}}{\partial t} - \sigma m^2 \tilde{v} = 0. \tag{38}$$

The solution to (38) is given by

$$\tilde{v}(m,t) = \tilde{v}^0(m)\mathbf{e}^{\sigma m^2 t}, \tag{39}$$

$$\tilde{v}^0(m) = \tilde{u}^0(m)\mathbf{e}^{-\sigma m^2 T}, \tag{40}$$

or,

$$\tilde{v}(m,t) = \tilde{u}(m,0)\mathbf{e}^{-\sigma m^2 (T-t)}. \tag{41}$$

In the Fourier variable, the exponential term does not improve regularity, it is just a multiplication. ■

5. CONCLUSION

The numerical integration of the Vlasov equation has been studied intensely during the recent years, since a knowledge of its nonlinear evolution is indispensable in the understanding of plasmas. A major difficulty encountered in these studies is the phase space filamentation of the distribution function. The filtering method introduced by Klimas is reminiscent of the Fokker–Planck term introduced in [7, 8] in the Fourier–Hermite method. However, the comparison is fallacious. The finite number of Hermite polynomials introduces a bouncing of the information and triggers instability. The Fokker–Planck term damps the high-order Hermite coefficients suppressing the instability but at the price of a modification of the physics of the problem. On the contrary, the method of Klimas involves a second-order term that does not change the problem; it can be compared with a heat equation which is forward in one variable and backward in the other variable. The filtering fixes the behavior of the initial conditions, behavior which is difficult to maintain as time goes on in a numerical simulation, unless one uses a Fourier–Fourier method. This was outlined by Klimas himself. However, the filtering is then just a multiplication which does not help numerically. The smallness at the border is just an artifact and tends to hide the reality of the approximation. It is important to point out that filamentation is a physical property, and that the splitting method does not trigger any numerical instability. Indeed, in any stable numerical method such as [17] or splines methods having a numerical viscosity, it is not necessary to introduce another one explicitly by a Gaussian filtering.

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